

Emittance Growth from the Thermalization of Space-Charge Nonuniformities*

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Abstract

Beams injected into a linear focusing channel typically have some degree of space-charge nonuniformity. In general, injected particle distributions with systematic charge nonuniformities are not equilibria of the focusing channel and launch a broad spectrum of collective modes. These modes can phase-mix and have nonlinear wave-wave interactions which, at high space-charge intensities, results in a relaxation to a more thermal-like distribution characterized by a uniform density profile. This thermalization can transfer self-field energy from the initial space-charge nonuniformity to the local particle temperature, thereby increasing beam phase space area (emittance growth). In this paper, we employ a simple kinetic model of a continuous focusing channel and build on previous work that applied system energy and charge conservation[1, 2] to quantify emittance growth associated with the collective thermalization of an initial azimuthally symmetric, rms matched beam with a radial density profile that is hollowed or peaked. This emittance growth is shown to be surprisingly modest even for high beam intensities with significant radial structure in the initial density profile.

1 INTRODUCTION

Experiments with high-current, heavy-ion injectors have observed significant space-charge nonuniformities emerging from the source. Sharp density peaks on the radial edge of beam have been measured, but the local incoherent thermal spread of particle velocities (i.e., the particle temperature) across the beam is anticipated to be fairly uniform since the beam is emitted from a constant temperature surface. When such a distribution is injected into a linear transport channel, it will be far from an equilibrium condition (i.e., particles out of local radial force balance), and a broad spectrum of collective modes will be launched.

The spatial average particle temperature of a heavy ion beam emerging from an injector is typically measured as several times what one would infer from the source thermal temperature ($\sim 0.1\text{eV}$) and subsequent beam envelope compressions, with $\bar{T}_x \sim 20\text{eV}$ where $\bar{T}_x \sim [\epsilon_x^2/(2R^2)]\mathcal{E}_b$. On the other hand, the radial change in potential energy from beam center to edge is $q\Delta\phi \sim 2.25\text{keV}$ for a beam with line-charge density $\lambda \sim 0.25\mu\text{C/m}$ ($\Delta\phi \sim \lambda/(4\pi\epsilon_0)$). If even a small fraction of such space-charge energy is thermalized during collective relaxation, large temperature and emittance increases can result.

In this paper, we employ conservation constraints to better estimate emittance increases from collective thermal-

ization of normal mode perturbations resulting from initial space-charge nonuniformities characteristic of intense beam injectors. Past studies have employed analogous techniques to estimate emittance increases resulting from the thermalization of initial rms mismatches in the beam envelope and space-charge nonuniformities associated with combining multiple beams and other processes[1, 2, 3].

2 THEORETICAL MODEL

We analyze an infinitely long, unbunched ($\partial/\partial z = 0$) non-relativistic beam composed of a single species of particles of mass m and charge q propagating with constant axial kinetic energy \mathcal{E}_b . Continuous radial focusing is provided by an external force that is proportional to the transverse coordinate \mathbf{x} , i.e., $\mathbf{F}_{\text{ext}} = -2\mathcal{E}_b k_{\beta 0}^2 \mathbf{x}$, where $k_{\beta 0} = \text{const}$ is the betatron wavenumber of particle oscillations in the applied focusing field. For simplicity, we neglect particle collisions and correlation effects, self-magnetic fields, and employ an electrostatic model and describe the transverse evolution of the beam as a function of axial propagation distance s in terms of a single-particle distribution function f that is a function of s , and the transverse position \mathbf{x} and angle $\mathbf{x}' = d\mathbf{x}/ds$ of a single particle. This evolution is described by the Vlasov equation[2],

$$\left\{ \frac{\partial}{\partial s} + \frac{\partial H}{\partial \mathbf{x}'} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}'} \right\} f(\mathbf{x}, \mathbf{x}', s) = 0, \quad (1)$$

where $H = \mathbf{x}'^2/2 + k_{\beta 0}^2 \mathbf{x}^2/2 + (q/2\mathcal{E}_b)\phi$ is the single-particle Hamiltonian and the self-field potential ϕ satisfies the Poisson equation (CGS units here and henceforth)

$$\nabla^2 \phi = -4\pi q \int d^2 x' f \quad (2)$$

subject to the boundary condition $\phi(r = r_p) = 0$ at the conducting pipe radius $r = |\mathbf{x}| = r_p = \text{const}$.

If no particles are lost in the beam evolution, the Vlasov-Poisson system possesses global constraints corresponding to the conservation of system charge (λ) and scaled energy (U) per unit axial length,

$$\begin{aligned} \lambda &= q \int d^2 x \int d^2 x' f = \text{const}, \\ U &= \frac{1}{2} \langle \mathbf{x}'^2 \rangle + \frac{k_{\beta 0}^2}{2} \langle \mathbf{x}^2 \rangle + \frac{q}{2\mathcal{E}_b \lambda} W = \text{const}. \end{aligned} \quad (3)$$

Here, $W \equiv \int d^2 x |\nabla \phi|^2/(8\pi)$ is the self-field energy of the beam per unit axial length and $\langle \xi \rangle \equiv (\int d^2 x \int d^2 x' \xi f)/(\int d^2 x \int d^2 x' f)$ is a transverse statistical average of ξ over the beam distribution f . Note that U includes both particle kinetic energy and the field energy of the applied and self-fields. These conservation laws follow directly from Eqs. (1)-(2) and provide powerful constraints on the nonlinear evolution of the system.

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Moment descriptions of the beam provide a simplified understanding of beam transport. For an azimuthally symmetric beam ($\partial/\partial\theta = 0$), a statistical measure of the beam edge radius $R \equiv 2\langle x^2 \rangle^{1/2}$ is employed. Note that R is the edge radius of a beam with uniformly distributed space-charge. Any axisymmetric solution to the Vlasov-Poisson system will be consistent with the rms envelope equation[1]

$$\frac{d^2 R}{ds^2} + k_{\beta 0}^2 R - \frac{Q}{R} - \frac{\epsilon_x^2}{R^3} = 0. \quad (4)$$

Here, $Q = q\lambda/\mathcal{E}_b = \text{const}$ is the self-field perveance and $\epsilon_x = 4[\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2]^{1/2}$ is an edge measure of the rms x -emittance of the beam and is a statistical measure of the beam area in x - x' phase-space (i.e., beam quality). For general distributions, ϵ_x is not constant and evolves according to the full Vlasov-Poisson system.

3 NONUNIFORM DENSITY PROFILE

We examine an beam with an azimuthally symmetric radial density profile $n = \int d^2 x' f$ given by

$$n(r) = \begin{cases} n_0 \left[1 - \frac{1-h}{h} \left(\frac{r}{r_b} \right)^p \right], & 0 \leq r \leq r_b, \\ 0, & r_b < r \leq r_p. \end{cases} \quad (5)$$

Here, r_b is the physical edge-radius of the beam, $n_0 = n(r=0)$ is the on-axis ($r=0$) beam density, and h and p are “hollowing” [$0 \leq h \leq \infty$, $h = n(r=r_b)/n(r=0)$, $p \geq 0$] and radial steepening parameters associated with the density nonuniformity. This density profile is illustrated in Fig. 1 for the steepening index $p = 2$ and hollowing factors $h = 1$ (uniform), $h = 1/2$ (hollowed), and $h = 2$ (peaked). The hollowing parameter h has range $0 \leq h < 1$ for an on-axis hollowed beam and $0 \leq 1/h < 1$ for an on-axis peaked beam. The limit $h \rightarrow 1$ corresponds to a uniform density beam and $h, 1/h \rightarrow 0$ correspond to hollowed and peaked beams with the density approaching zero on-axis and at the beam edge ($r = r_b$), respectively. For large steepening index $p \gg 1$, the density gradient will be significant only near the radial edge of the beam ($r \simeq r_b$), and the density is uniform for $h = 1$ regardless of p .

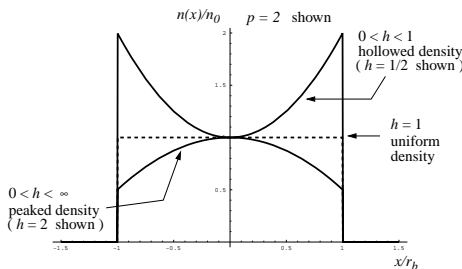


Figure 1: Uniform, hollowed, and peaked density profiles.

The beam line-charge density (λ) and rms edge-radius (R) are related to the parameters in Eq. (5) by

$$\lambda = \int d^2 x n = \pi q n_0 r_b^2 \left[\frac{(ph+2)}{(p+2)h} \right] \quad (6)$$

$$R = 2\langle x^2 \rangle^{1/2} = \sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}} r_b$$

Using these expressions, the Poisson equation (2) can be solved for the potential ϕ corresponding to the density profile (5) and used to calculate the self-field energy W as

$$W = \lambda^2 \left\{ \frac{1}{(ph+2)^2} \left[\frac{(p+2)^2 h^2}{4} + \frac{2(1-h)^2}{p+2} + \frac{4(p+2)h(1-h)}{p+4} \right] + \ln \left[\sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}} \frac{r_p}{R} \right] \right\}. \quad (7)$$

It is convenient to *define* an average phase advance parameter σ for the density profile (5) in terms of an envelope matched ($R' = 0 = R''$), rms equivalent beam with uniform density ($h = 1$) and the same perveance (Q) and emittance (ϵ_x) as the (possibly mismatched) beam with a nonuniform density profile ($h \neq 1$). Denoting the phase advance per unit axial length of transverse particle oscillations in the matched equivalent beam in the presence and absence of space-charge by σ and σ_0 , we adapt a normalized space-charge parameter $\sigma/\sigma_0 \equiv \sqrt{k_{\beta 0}^2 - Q/R^2}/k_{\beta 0}$. The limits $\sigma/\sigma_0 \rightarrow 0$ and $\sigma/\sigma_0 \rightarrow 1$ correspond to a cold, space-charge dominated beam and a warm, kinetic dominated beam, respectively. Note that this measure applies only in an equivalent beam sense. In general, distributions f consistent with the density profile (5) will not be equilibria ($d/ds \neq 0$) of the transport channel and will evolve leaving σ ill defined.

4 EMITTANCE GROWTH

We consider an initial beam distribution f with a density profile given by Eq. (5) and an *arbitrary* “momentum” distribution in x' . Such an initial distribution is not, in general, an equilibrium of the focusing channel and a spectrum of collective modes will be launched (depending on the full initial phase-space structure of f). These modes will phase-mix, have nonlinear wave-wave interactions, etc., driving relaxation processes that have been observed in PIC simulations to cause the beam space-charge distribution to become more uniform for the case of high beam intensities. The conservation constraints (3) are employed to connect the parameters of an initial (subscript i), nonuniform density beam with $h \neq 0$ with those of a final (subscript f), azimuthally symmetric and rms envelope matched beam ($R'_f = 0 = R''_f$) with uniform density ($h = 1$).

Employing Eqs. (4)-(7), conservation of charge ($\lambda_i = \lambda_f \equiv \lambda$) and system energy ($U_i = U_f$) can be combined into an single equation of constraint expressible as

$$\frac{(R_f/R_i)^2 - 1}{1 - (\sigma_i/\sigma_0)^2} + \frac{p(1-h)[4+p+(3+p)h]}{(p+2)(p+4)(2+ph)^2} - \ln \left[\sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}} \frac{R_f}{R_i} \right] = \frac{\mathcal{E}_b}{2q\lambda} (R_i R'_i)' \quad (8)$$

Here, h and p are the hollowing factor and index of the initial density profile, σ_i/σ_0 is the initial space-charge intensity, and $[\mathcal{E}_b/(2q\lambda)](R_i R'_i)'$ is a parameter that measures the initial envelope mismatch of the beam. This nonlinear constraint equation can be solved numerically for fixed $h, p, \sigma_i/\sigma_0$ and $[\mathcal{E}_b/(2q\lambda)](R_i R'_i)'$ to determine the ratio of

final to initial rms radius of the beam (R_f/R_i). Employing the envelope equation (4), the ratio of final to initial beam emittance is expressible as

$$\frac{\epsilon_{xf}}{\epsilon_{xi}} = \frac{R_f}{R_i} \sqrt{\frac{(R_f/R_i)^2 - [1 - (\sigma_i/\sigma_0)^2]}{(\sigma_i/\sigma_0)^2 - R_i''/(k_{\beta 0}^2 R_i)}}. \quad (9)$$

Eqs. (8) and (9) allow analysis of emittance growth from the thermalization of initial space-charge nonuniformities.

We numerically solve Eqs. (8) and (9) to plot (Fig. 2) the growth in rms beam radius (R_f/R_i) and emittance ($\epsilon_{xf}/\epsilon_{xi}$) due to the relaxation of an initial rms matched beam ($R_i' = 0 = R_i''$) with nonuniform hollowed and peaked density profiles to a final uniform, matched profile. Final to initial beam ratios are shown for hollowing index of $p = 2$ and are plotted versus the “hollowing factors” h (hollow initial density) and $1/h$ (peaked initial density) for families of σ_i/σ_0 ranging from $\sigma_i/\sigma_0 \rightarrow 0$ to $\sigma_i/\sigma_0 \rightarrow 1$. Growths are larger for the initially hollowed profile than the peaked profile and increase with stronger space-charge (smaller σ_i/σ_0). However, the change in rms radius (R_f/R_i) is small in all cases, even for strong space-charge with strong hollowing ($h \rightarrow 0$) and peaking ($1/h \rightarrow 0$) parameters. Moreover, the increases in beam emittance ($\epsilon_{xf}/\epsilon_{xi}$) are surprisingly modest (factor of 2 and less) for intense beam parameters with $\sigma_i/\sigma_0 \sim 0.1$ and greater. At fixed σ_i/σ_0 and increasing steepening factor p ,

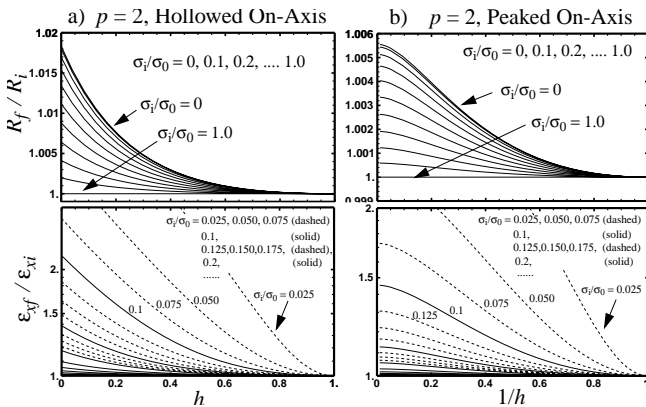


Figure 2: Ratio of final to initial rms beam size (R_f/R_i) and emittance ($\epsilon_{xf}/\epsilon_{xi}$) versus h (a, hollowed beam) and $1/h$ (b, peaked beam).

similar modest growth factors are seen for hollowed beams for all but the most extreme hollowing factors ($h \sim 0.1$ and less), and as expected, much less growth is seen for peaked beams (closer to uniform).

5 DISCUSSION

The modest emittance growth at high beam intensities can be understood as general beyond the specific model employed. Even for significant increases in emittance ϵ_x , the rms matched beam size is given to a good approximation by the envelope equation (4) with the emittance term neglected. In this case, $R \simeq \sqrt{Q}/k_{\beta 0} = \text{const}$ during the beam evolution and hence $R_f \simeq R_i$. Employing the method of Lagrange multipliers, the free electrostatic energy of the system at fixed rms radius (R) and line-charge

(λ) can be expressed as $F = W - \int d^2x (\mu_1 r^2 + \mu_2)n$ with $\mu_{1,2} = \text{const}$. Taking variations $\delta\phi$ subject to the Poisson equation (2), one obtains to arbitrary order in $\delta\phi$,

$$\delta F = \int d^2x (q\phi - \mu_1 \mathbf{x}^2 - \mu_2)\delta n + \int d^2x \frac{|\nabla\delta\phi|^2}{8\pi}. \quad (10)$$

Thus, constrained extrema of F satisfy $q\phi = \mu_1 \mathbf{x}^2 + \mu_2$, corresponding to a uniform density beam centered on-axis. Variations about this extremum satisfy $\delta F > 0$ and are second order in $\delta\phi$. Thus, the available electrostatic energy for thermalization induced emittance increase is modest for any smooth density profile. This can be demonstrated for our specific example using equation (7) to plot $\Delta F = W_i - W_f$ with $R_i = R_f$ versus h and $1/h$ for $p = 2, 8$ (Fig. 3).

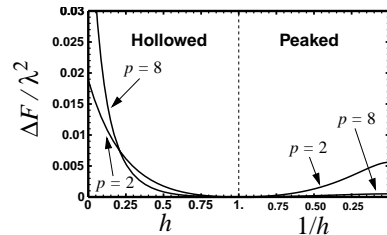


Figure 3: Free energy versus hollowing factors h and $1/h$.

It has been shown that the rms beam size and emittance undergo very small *decreases* on relaxation from a uniform density beam to thermal equilibrium over the full range of σ_i/σ_0 (Min [$\epsilon_{xf}/\epsilon_{xi}$] $\simeq 0.97$ at $\sigma_i/\sigma_0 \simeq 0.45$)[4]. Thus if one views the relaxation as a multi-step procedure using the conservation constraints to connect the initial nonuniform profile to a uniform profile and then a thermal profile, any emittance growth will be experienced in the first step. This result together with the variational argument above show that the emittance growth results presented should be relatively insensitive to the form of the final distribution.

Finally, caveats should be given for validity of the theory. First, the model assumes no generation of halo in the final state and that the initial nonuniform beam can be perfectly rms envelope matched. Initial mismatches can lead to halo production and provide a large source of free energy which, if thermalized, can lead to significant emittance growth[1]. Also, although the velocity space distribution is arbitrary in the present theory, choices that project onto broader spectrums of modes will more rapidly phase mix and thermalize. Small applied nonlinear fields tend to enhance this relaxation rate. Initial simulation results in a full AG lattice are consistent with model predictions presented here and will be presented in future work.

6 REFERENCES

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